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Further results on McMahon's asymptotic approximations

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Abstract. We consider the sum of the first two or three terms from the McMahon asymptotic expansion of the zeros of the cylinder function $C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$, $0 \leqslant \alpha < \pi$ and study when this sum represents as an upper or lower bound for the corresponding zero. The results established extend—in particular—the case of the zeros of $J_\nu(x)$, when we recover the inequalities found by Förster and Petras (Förster K J and Petras K 1993 *ZAMM* **73** 232–6) for $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$. Our approach is based on a Sturmian comparison theorem discussed in section 2.

1. Introduction

Let $j_{\nu\kappa}$, $\kappa = k - \alpha/\pi$, $k = 1, 2, \ldots$, denote the kth positive zero $c_{\nu k}$ of the cylinder function

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha \qquad 0 \leqslant \alpha < \pi$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively. For more information on the notation $j_{\nu\kappa}$ we refer to [2]. In particular, we have

$$j_{1/2,\kappa} = \kappa\pi \qquad \text{for} \quad \kappa > 0 \qquad j_{-1/2,\kappa} = (\kappa - \tfrac{1}{2})\pi \qquad \text{for} \quad \kappa > \tfrac{1}{2}. \tag{1.1}$$

When ν fixed and κ large, McMahon established the following asymptotic result [8, p 506]:

$$j_{\nu\kappa} = \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} - \frac{4(\mu-1)(7\mu-31)}{3(8\beta_{\nu\kappa})^3} + \mathrm{O}(\beta_{\nu\kappa}^{-5}) \qquad \kappa \to \infty \tag{1.2}$$

where

$$\beta_{\nu\kappa} = \left(\kappa + \frac{\nu}{2} - \frac{1}{4}\right)\pi \qquad \mu = 4\nu^2.$$

Further terms from the McMahon asymptotic formula can be found, e.g., in [8, p 506]. Concerning the first term $\beta_{\nu\kappa}$ of (1.2), it is proved in [6] that

$$j_{\nu\kappa} \geqslant \beta_{\nu\kappa} \qquad \text{for} \quad |\nu| \leqslant \tfrac{1}{2} \tag{1.3a}$$

$$j_{\nu\kappa} \leqslant \beta_{\nu\kappa} \qquad \text{for} \quad |\nu| \geqslant \tfrac{1}{2} \tag{1.3b}$$

where equalities occur if and only if $|\nu| = \frac{1}{2}$, in accordance with (1.1). The inequality (1.3*a*) is already presented in [8, p 506].

In the whole paper we make the restriction on κ that $\beta_{\nu\kappa} > 0$. This is a real restriction only in the case when $|\nu| < \frac{1}{2}$ due to (1.3*b*).

In the particular case of the Bessel function $J_\nu(x)$, Hethcote [5] established the more informative result

$$j_{\nu k} \leqslant \beta_{\nu k} - \frac{\mu - 1}{8\beta_{\nu k}} \qquad 0 \leqslant \nu \leqslant \tfrac{1}{2} \qquad k = 1, 2, \ldots, \tag{1.4}$$

while, for $|\nu| \leqslant \frac{1}{2}$ and $k = 1, 2, \ldots$, Förster and Petras [3] obtained the inequalities

$$\beta_{\nu k} - \frac{\mu - 1}{8\beta_{\nu k}} - \frac{4(\mu - 1)(7\mu - 31)}{3(8\beta_{\nu k})^3} \leqslant j_{\nu k} \leqslant \beta_{\nu k} - \frac{\mu - 1}{8\beta_{\nu k}} \tag{1.5}$$

where the upper bound is the same as in (1.4), but extended also to negative values of ν.

In this paper we prove two theorems. The first is concerned with the two-term approximation of the asymptotic expansion (1.2). The second one gives inequalities for $j_{\nu\kappa}$ using the three-term approximation. The results are formulated as follows.

Theorem 1. *For $j_{\nu\kappa}$ defined above, the following inequalities hold:*

$$j_{\nu\kappa} < \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} \qquad \text{for} \quad |\nu| < \frac{1}{2} \qquad \text{provided} \quad j_{\nu\kappa} \geqslant 2\sqrt{\frac{1}{8} - \frac{1}{2}\nu^2} \tag{1.6a}$$

$$j_{\nu\kappa} > \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} \qquad \text{for} \quad \frac{1}{2} < |\nu| < \sqrt{\frac{31}{28}}$$

$$\text{provided} \quad j_{\nu\kappa} \geqslant \frac{503 + 196\nu^2}{24\sqrt{70}}\sqrt{\frac{4\nu^2 - 1}{31 - 28\nu^2}} \tag{1.6b}$$

$$j_{\nu\kappa} < \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} \qquad \text{for} \quad |\nu| \geqslant \sqrt{\frac{31}{28}} \qquad \kappa > 0 \qquad \nu + \kappa > 0. \tag{1.6c}$$

Theorem 2. *Let $j_{\nu\kappa}$ be defined as above. Then*

$$j_{\nu\kappa} > F(\beta_{\nu\kappa}) \qquad \text{for} \quad |\nu| < \tfrac{1}{2} \qquad \text{provided} \quad \beta_{\nu\kappa} \geqslant \sqrt{\tfrac{5}{8} - \tfrac{1}{2}\nu^2} \tag{1.7}$$

$$j_{\nu\kappa} < F(\beta_{\nu\kappa}) \qquad \text{for} \quad \frac{1}{4} < \nu^2 < \frac{31 + 4\sqrt{78}}{28} \qquad j_{\nu\kappa} \geqslant F\left(\frac{1}{2}\nu^2 + \frac{3}{8}\right) \tag{1.8a}$$

$$j_{\nu\kappa} < F(\beta_{\nu\kappa}) \qquad \text{for} \quad \nu^2 \geqslant \frac{31 + 4\sqrt{78}}{28} \qquad \kappa > 0 \qquad \nu + \kappa > 0 \tag{1.8b}$$

where

$$F(\beta) = \beta - \frac{\mu - 1}{8\beta} - \frac{4(\mu - 1)(7\mu - 31)}{3(8\beta)^3}.$$

The statements in theorems 1, 2 are not sharp. Numerical calculation shows, for example, that (1.6*a*) always holds when $\nu = 0$ and $\beta_{0,\kappa} > 0$. The restrictions on κ (actually on $j_{\nu\kappa}$ or $\beta_{\nu\kappa}$) are consequences of the method used. The method is an application of a Sturmian comparison theorem which will be proved in the next section.

The results established in this paper are general in the sense that they hold for *every* zero of *every* cylinder function subject to a reasonable lower bound on the zeros. The restrictions in (1.6*a*), (1.6*c*), (1.7), (1.8*a*), (1.8*b*) are satisfied for all $\kappa \geqslant 1$ when $\nu \geqslant 0$ because we have the inequality $j_{\nu\kappa} \geqslant j_{\nu 1} \geqslant \nu + j_{01}$ and $j_{01} = 2.40, \ldots$. For negative ν we can use the identity [1]

$$j_{\nu\kappa} = j_{-\nu,\nu+\kappa} \qquad (\nu > -\kappa).$$

(In addition, we can observe the fact that we also have the equality $\beta_{\nu\kappa} = \beta_{-\nu,\nu+\kappa}$.)

Finally, we want to refer to a recent paper [4] of Gatteschi and Giordano who give error bounds for the two- and three-term approximations of McMahon's formula in the particular case of the zeros $j_{\nu k}$. Their bounds are quite sharp when $-\frac{1}{2} < \nu < \frac{1}{2}$, but when $\nu > \frac{1}{2}$ (the case $\nu < -\frac{1}{2}$ is not discussed there), they make the restriction $j_{\nu k} > (2\nu + 1)(2\nu + 3)/\pi$, which excludes several zeros $j_{\nu k}$, when ν is large but k is not sufficiently large. This happens because for fixed k the asymptotic formula [8, p 521]

$$j_{\nu k} = \nu + a_k \nu^{1/3} + \mathrm{O}(\nu^{-1/3}) \qquad \nu \to \infty$$

where a_k is independent of ν, holds.

We conclude this section by mentioning that the function $\sqrt{x}C_\nu(x)$ is a solution of the linear second-order differential equation [8, p 117]

$$y'' + \left(1 + \frac{2c}{x^2}\right) y = 0 \tag{1.9}$$

where

$$2c = \tfrac{1}{4} - \nu^2. \tag{1.10}$$

2. A Sturmian comparison theorem

The main result of this section is a Sturmian comparison theorem which will be used in the proof of theorems 1, 2. First we make a simple observation which will be written as the following lemma.

Lemma 1. *Let the function $\varphi(x)$ be three-times differentiable on some interval I with $\varphi'(x) > 0$. Then the functions*

$$z_1(x) = \rho(x) \sin \varphi(x) \qquad \textit{and} \qquad z_2(x) = \rho(x) \cos \varphi(x)$$

are linearly independent solutions of the differential equation

$$z'' + q(x) z = 0 \tag{2.1}$$

where

$$\rho(x) = [\varphi'(x)]^{-1/2} \qquad \textit{and} \qquad q(x) = [\varphi'(x)]^2 - \frac{\rho''(x)}{\rho(x)}. \tag{2.2}$$

Proof. The statement can be easily verified by direct substitution of the functions $\rho \sin \phi$ and $\rho \cos \phi$ into the differential equation (2.1). □

Now we formulate our comparison theorem.

Lemma 2. *Let $y(x)$ be a solution of the differential equation*

$$y'' + \tilde{q}(x) y = 0 \qquad x \in \tilde{I} = (\tilde{a}, \infty) \tag{2.3}$$

with consecutive zeros at $\tilde{x}_k = (k-1)\pi + \alpha_k$ in $\tilde{I}$, $k = 1, 2, \ldots$, such that

$$\lim_{k \to \infty} \alpha_k = \alpha \text{ (finite)}. \tag{2.4}$$

Let $\varphi(x)$ satisfy the same properties as in lemma 1 with $I = (a, \infty)$ and also the condition

$$\lim_{x \to \infty} [\varphi(x) - x] = 0. \tag{2.5}$$

Suppose that

$$\tilde{q}(x) \underset{(>)}{<} q(x) \qquad \textit{for} \quad x \geqslant \hat{a} \geqslant \max\{a, \tilde{a}\} \tag{2.6}$$

where $q(x)$ *is the same as in (2.2). Let* k_1 *be an integer such that* $\tilde{x}_{k_1} \geqslant \hat{a}$. *Then let*

$$\hat{k} = \begin{cases} k_1 & \text{if} \quad \tilde{q}(x) < q(x) \quad \text{for} \quad x \geqslant \hat{x} \\ \max\{k_0, k_1\} & \text{if} \quad \tilde{q}(x) > q(x) \quad \text{for} \quad x \geqslant \hat{x} \end{cases}$$

where k_0 *satisfies the inequality* $(k_0 - 1)\pi + \alpha \geqslant \varphi(a)$.

Define the values $x_{\hat{k}}, x_{\hat{k}+1}, \ldots$ *by*

$$\varphi(x_k) = (k-1)\pi + \alpha \qquad \textit{for} \quad k \geqslant \hat{k}. \tag{2.7}$$

Then $\{x_k\}_{k=\hat{k}}^{\infty}$ *are consecutive zeros of* $z(x) = \cos\alpha\, z_1(x) - \sin\alpha\, z_2(x)$ *satisfying the relations*

$$\tilde{x}_k \underset{(>)}{<} x_k \qquad \textit{for} \quad k = \hat{k}, \hat{k}+1, \ldots \qquad \textit{and} \qquad \lim_{k\to\infty} (x_k - \tilde{x}_k) = 0. \tag{2.8}$$

The solution $z(x)$ *is unique up to a constant multiple.*

Proof. For $k = 1, 2, \ldots$ consider the solution $Z_k(x) = \rho(x)\sin[\varphi(x) - \varphi(\tilde{x}_k)]$ of (2.1), and observe that $y(x)$ and $Z_k(x)$ have a common zero at $x = \tilde{x}_k$. Suppose that $\tilde{q}(x) < q(x)$ holds. Then equation (2.1) is a Sturmian majorant of (2.3). Consequently, by the standard Sturm comparison theorem [7, p 19], $Z_k(x)$ has at least one zero, say ξ, in the interval $(\tilde{x}_k, \tilde{x}_{k+1})$ with $\varphi(\xi) - \varphi(\tilde{x}_k) = \pi$ and $\xi < \tilde{x}_{k+1}$. Recalling that $\varphi(x)$ is increasing, we get

$$\varphi(\tilde{x}_k) + \pi = \varphi(\xi) < \varphi(\tilde{x}_{k+1}).$$

Therefore

$$\varphi(\tilde{x}_k) - (k-1)\pi < \varphi(\tilde{x}_{k+1}) - k\pi.$$

Hence the sequence $\{\varphi(\tilde{x}_k) - (k-1)\pi\}_{k=1}^{\infty}$ is increasing and by (2.4), (2.5)

$$\lim_{k\to\infty} [\varphi(\tilde{x}_k) - (k-1)\pi] = \lim_{k\to\infty} [\varphi(\tilde{x}_k) - \tilde{x}_k] + \lim_{k\to\infty} (\tilde{x}_k - (k-1)\pi) = \alpha.$$

Thus we obtain

$$\varphi(\tilde{x}_k) - (k-1)\pi < \alpha$$

or by (2.7) $\varphi(\tilde{x}_k) < (k-1)\pi + \alpha = \varphi(x_k)$, consequently $\tilde{x}_k < x_k$.

Since $\tilde{x}_k = (k-1)\pi + \alpha_k$, $\varphi(x_k) = (k-1)\pi + \alpha$, the limit relation $\lim_{k\to\infty}(x_k - \tilde{x}_k) = 0$ follows from (2.4), (2.5).

The case $\tilde{q}(x) > q(x)$ can be dealt in a similar way, which completes the proof of lemma 2. □

The application of lemma 2 might involve a difficult calculation particularly in justifying the inequality in (2.6). Sometimes it is preferable to work with the inverse function $f = \varphi^{-1}$ of φ. Thus we introduce the function

$$x = f(\varphi) > \tilde{a} \qquad \text{with} \quad f'(\varphi) > 0 \qquad \text{for} \quad \varphi > \tilde{\varphi}. \tag{2.9}$$

Differentiation of $f(\varphi)$ in (2.9) with respect to x gives

$$1 = f'(\varphi)\varphi'(x)$$

therefore

$$\varphi'(x) = \frac{1}{f'(\varphi)}.$$

Then, by (2.2)

$$\rho^2(x) = f'(\varphi) \tag{2.10}$$

and differentiation of $\rho^2(x)$ with respect to x gives

$$2\rho(x)\rho'(x) = \frac{f''(\varphi)}{f'(\varphi)}$$

hence

$$\frac{\rho'(x)}{\rho(x)} = \frac{1}{2\rho^2(x)}\frac{f''(\varphi)}{f'(\varphi)} = \frac{1}{2}\frac{f''(\varphi)}{[f'(\varphi)]^2}.$$

Further differentiation yields

$$\begin{aligned}\frac{\rho''(x)}{\rho(x)} &= \left[\frac{\rho'(x)}{\rho(x)}\right]^2 + \frac{1}{2}\left[\frac{f'''(\varphi)}{[f'(\varphi)]^2} - 2\frac{[f''(\varphi)]^2}{[f'(\varphi)]^3}\right]\varphi'(x)\\ &= \frac{1}{4}\left[\frac{f''(\varphi)}{[f'(\varphi)]^2}\right]^2 + \frac{1}{2}\frac{f'(\varphi)f'''(\varphi)}{[f'(\varphi)]^4} - \frac{[f''(\varphi)]^2}{[f'(\varphi)]^4}\\ &= \frac{2f'(\varphi)f'''(\varphi) - 3[f''(\varphi)]^2}{4[f'(\varphi)]^4}.\end{aligned}$$

Now inequality (2.6) takes the form

$$\tilde{q}[f(\varphi)] \underset{(>)}{<} \frac{1}{[f'(\varphi)]^2} - \frac{2f'(\varphi)f'''(\varphi) - 3[f''(\varphi)]^2}{4[f'(\varphi)]^4} \qquad \text{for} \quad \varphi > \hat{\varphi} \tag{2.11}$$

where

$$\hat{a} = \max\{f(\hat{\varphi}), f(\tilde{\varphi})\} \tag{2.12}$$

and (2.8) becomes

$$\tilde{x}_k \underset{(>)}{<} f((k-1)\pi + \alpha). \tag{2.13}$$

In our applications the function $\tilde{q}(x)$ will be the coefficient in (1.9), i.e. $\tilde{q}(x) = 1 + 2c/x^2$, and $\tilde{x}_k = c_{\nu k} = j_{\nu\kappa}$, consequently $\tilde{a} = 0$. The consecutive zeros of $C_\nu(x)$ are $\tilde{x}_1 = j_{\nu,\kappa}$, $\tilde{x}_2 = j_{\nu,\kappa+1}$, $\tilde{x}_3 = j_{\nu,\kappa+2}, \ldots$, hence for the α in (2.4) we have by (1.2) $\alpha = \beta_{\nu\kappa}$.

Remark 2. In the particular case when $\varphi(x) = x$, we recover the results (1.3*a*), (1.3*b*) because $q(x) = 1$ by (2.2) and application of lemma 2 with $\tilde{q}(x) = 1 + 2c/x^2$ for $c > 0$ (or $c < 0$) gives the already known bounds.

3. Two-term approximation

Proof of theorem 1. We choose

$$x = f(\varphi) = \varphi + \frac{c}{\varphi}.$$

Then

$$\varphi(x) = \frac{x + \sqrt{x^2 - 4c}}{2}$$

and we have

$$a = \begin{cases} 2\sqrt{c} & \text{if} \quad c \geqslant 0 \\ -\infty & \text{if} \quad c < 0 \end{cases} \qquad \tilde{\varphi} = \varphi(a) = \begin{cases} \sqrt{c} & \text{if} \quad c \geqslant 0 \\ 0 & \text{if} \quad c < 0 \end{cases}$$

and condition (2.5) of lemma 2 is satisfied. We will show that (2.6) is valid with $\hat{\varphi} = \sqrt{|c|}$—except the case (b). To this end we consider the difference

$$\tilde{q}[f(\varphi)] - \frac{1}{[f'(\varphi)]^2} + \frac{2f'(\varphi)f'''(\varphi) - 3[f''(\varphi)]^2}{4[f'(\varphi)]^4} = c\frac{P(c,\varphi^2)}{4(\varphi^2 - c)^4(\varphi^2 + c)^2} \tag{3.1}$$

where

$$P(c,z) = c^5 - 3c^2z^2 - 10c^3z^2 - 6cz^3 + 16c^2z^3 - 3z^4 - 7cz^4.$$

In the case $|\nu| < \frac{1}{2}$, we get by (1.10) that $\frac{1}{8} \geqslant c > 0$ and

$$P(c, c+s) = -12c^4 - 36c^3s - (39c^2 + 4c^3)s^2 - (18c + 12c^2)s^3 - (3 + 7c)s^4.$$

Clearly, all the coefficients of this polynomial (of the variable s) are negative for $0 < c \leqslant \frac{1}{8}$. Therefore $P(c, c+s) < 0$ for $s \geqslant 0$. Thus equation (2.1) is a Sturmian majorant of (1.9), i.e. $\tilde{q}(x) < q(x)$ for $x \geqslant \hat{x} = f(\hat{\varphi}) = 2\sqrt{c}$ and an application of lemma 2 gives (1.6a), i.e. part (a) of theorem 1.

For the proof of (1.6b) we make the following substitution:

$$z = \frac{90}{7}\frac{c}{-3-7c} + s \qquad -\frac{3}{7} < c < 0$$

in the polynomial $P(c, z)$ above. We get

$$\begin{aligned} 2401(3+7c)^3 P\left(c, \frac{90}{7}\frac{c}{-3-7c} + s\right) \\ &= c^4(-12\,854\,700 - 3947\,391c + 100\,842c^2 + 117\,649c^3) \\ &\quad +1260c^3(10\,971 + 17\,619c + 3430c^2)s \\ &\quad -49c^2(3+7c)(37\,701 + 32\,739c + 3430c^2)s^2 \\ &\quad +686c(3+7c)^2(159 + 56c)s^3 - 2401(3+7c)^3s^4. \end{aligned}$$

On the right-hand side the coefficient of s^i is negative for $i = 0, 1, \ldots, 4$ hence in (3.1) $cP(c, \varphi^2)$ is positive for $\varphi \geqslant \hat{\varphi}$, where $\hat{\varphi} = \sqrt{90c/(7(-3-7c))}$, implying that equation (1.9) is a Sturmian majorant of (2.1). An application of lemma 2 gives the desired result (1.6b) with $\hat{a} = f(\hat{\varphi}) = \frac{69-49c}{3\sqrt{70}}\sqrt{\frac{-c}{3+7c}}$. By (1.3$b$) the restriction $\alpha = \beta_{\nu\kappa} > 0 = \varphi(a)$ is clearly satisfied.

It remains to prove (1.6c). To this end we shall make use of the subtitution

$$z = -c + s \qquad c \leqslant -\tfrac{3}{7}$$

and we find

$$P(c, -c+s) = -(3+7c)s^4 + (6c + 44c^2)s^3 - (3c^2 + 100c^3)s^2 + 96c^4s - 32c^5$$

where all the coefficients are positive. This shows that the function in (3.1) is negative for $\varphi \geqslant \hat{\varphi} = \sqrt{|c|}$ and that $\tilde{q}(x) < q(x)$ for $x > f(\hat{\varphi}) = 0$. Again, an application of lemma 2 gives (1.6c). This completes the proof of theorem 1 on the two-term approximation of the McMahon formula. □

4. Three-term approximation

Proof of theorem 2. Let $f(\varphi) = F(\varphi)$, i.e.

$$x = f(\varphi) = \varphi + \frac{c}{\varphi} - \frac{3c + 7c^2}{6\varphi^3}.$$

Now the inverse function $\varphi = \varphi(x)$ of $x = f(\varphi)$ is not at our disposal as it was in the case of two-term approximation. So we have to enter in the investigation of the behaviour of the function $f(\varphi)$. By (2.9) we find

$$\tilde{\varphi}^2 = \begin{cases} \dfrac{-c + \sqrt{(6c + 17c^2)/3}}{2} & \text{for} \quad c \geqslant 0 \\ \dfrac{c + \sqrt{-6c - 13c^2}}{2} & \text{for} \quad -\dfrac{6}{17} < c < 0 \\ \dfrac{-c + \sqrt{(6c + 17c^2)/3}}{2} & \text{for} \quad c \leqslant -\dfrac{6}{17}. \end{cases}$$

By (2.11) we have to deal with the expression

$$\tilde{q}[f(\varphi)] - \frac{1}{[f'(\varphi)]^2} + \frac{2f'(\varphi)f'''(\varphi) - 3[f''(\varphi)]^2}{4[f'(\varphi)]^4} = c\frac{Q(c, \varphi^2)}{(3c + 7c^2 - 6c\varphi^2 - 6\varphi^4)^2(3c + 7c^2 - 2c\varphi^2 + 2\varphi^4)^4} \tag{4.1}$$

where

$$\begin{aligned} cQ(c, z) = {} & 729c^6 + 10\,206c^7 + 59\,535c^8 + 185\,220c^9 + 324\,135c^{10} \\ & +302\,526c^{11} + 117\,649c^{12} \\ & -(4860c^6 + 56\,700c^7 + 264\,600c^8 + 617\,400c^9 + 720\,300c^{10} + 336\,140c^{11})z \\ & +(-972c^5 + 1296c^6 + 65\,016c^7 + 289\,296c^8 + 498\,036c^9 + 307\,328c^{10})z^2 \\ & +(-648c^4 + 1728c^5 + 34\,992c^6 + 106\,176c^7 + 107\,800c^8 + 21\,952c^9)z^3 \\ & -(864c^4 + 26\,568c^5 + 146\,376c^6 + 293\,048c^7 + 198\,744c^8)z^4 \\ & +(4752c^3 + 61\,776c^4 + 289\,296c^5 + 578\,480c^6 + 418\,656c^7)z^5 \\ & -(11\,664c^3 + 100\,224c^4 + 276\,912c^5 + 248\,064c^6)z^6 \\ & +(-11\,232c^2 - 24\,192c^3 + 79\,200c^4 + 173\,824c^5)z^7 \\ & +(22\,896c^2 + 35\,136c^3 - 44\,400c^4)z^8 + (8640c + 19\,584c^2 + 15\,936c^3)z^9. \end{aligned}$$

We distinguish two cases: (*a*) $0 \leqslant c \leqslant \frac{1}{8}$, (*b*) $c < 0$. In the first case we consider the polynomial

$$\begin{aligned} Q(c, c + \tfrac{1}{2} + s) = {} & \tfrac{1}{16}(270 + 5499c + 52\,266c^2 + 313\,233c^3 \\ & +1345\,148c^4 + 4392\,224c^5 + 11\,037\,272c^6 + 20\,895\,304c^7 \\ & +28\,536\,368c^8 + 26\,301\,648c^9 + 14\,562\,304c^{10} + 3647\,952c^{11}) \\ & +\tfrac{1}{4}(1215 + 23\,004c + 198\,945c^2 + 1052\,538c^3 \\ & +3844\,000c^4 + 10\,270\,636c^5 + 20\,383\,864c^6 + 29\,365\,912c^7 \\ & +28\,873\,904c^8 + 17\,187\,088c^9 + 4637\,136c^{10})s \\ & +(2430 + 42\,174c + 326\,709c^2 + 1499\,406c^3 \\ & +4557\,705c^4 + 9657\,892c^5 + 14\,417\,868c^6 \\ & +14\,649\,552c^7 + 9128\,780c^8 + 2627\,120c^9)s^2 \\ & +2(5670 + 88\,641c + 602\,586c^2 + 2340\,903c^3 \\ & +5737\,588c^4 + 9200\,844c^5 + 9557\,600c^6 + 5940\,772c^7 + 1694\,160c^8)s^3 \\ & +2(17\,010 + 234\,171c + 1358\,604c^2 + 4310\,733c^3 \\ & +8103\,146c^4 + 9085\,192c^5 + 5700\,028c^6 + 1578\,676c^7)s^4 \end{aligned}$$

$$+8(8505 + 99\,981c + 475\,839c^2 + 1166\,727c^3 + 1541\,184c^4$$
$$+1025\,974c^5 + 262764c^6)s^5$$
$$+16(5670 + 54\,432c + 200\,115c^2 + 344\,940c^3 + 267\,467c^4 + 66\,508c^5)s^6$$
$$+32(2430 + 17\,739c + 45\,594c^2 + 45\,669c^3 + 12260c^4)s^7$$
$$+48(810 + 3933c + 5898c^2 + 2063c^3)s^8 + 192(45 + 102c + 83c^2)s^9.$$

We find that all the coefficients of s^i, $i = 0, 1, \ldots, 9$, are positive for $0 \leqslant c \leqslant \frac{1}{8}$. Thus $Q(c, \varphi^2)$ is positive for $\varphi^2 \geqslant c + \frac{1}{2} \geqslant \tilde{\varphi}^2$ and $\tilde{q}(x) > q(x)$. We can apply lemma 2 and by (2.13) we obtain (1.7). □

For $c < 0$ we use the substitution

$$\varphi^2 = -c + s + \tfrac{1}{2}$$

and we have

$$Q(c, -c + s + \tfrac{1}{2}) = \sum_{j=0}^{9} A_j(c)s^j.$$

Now we have to show that all the coefficients of s^i are positive. Making use of the symbolic programing of *Mathematica*, we find

$$A_0(c) = \tfrac{1}{16}(-5767\,344c^{11} + 44\,872\,320c^{10} - 9006\,256c^9 + 6227\,056c^8$$
$$-2272\,888c^7 + 274\,488c^6 + 399\,264c^5 - 148\,756c^4$$
$$-30\,711c^3 + 23\,754c^2 - 4221c + 270)$$
$$A_1(c) = 5207\,092c^{10} - 9543\,052c^9 + 3479\,788c^8 - 1665\,898c^7$$
$$+437\,934c^6 + 137\,455c^5 - 118\,304c^4 - \tfrac{1}{2}9\,915c^3$$
$$+\tfrac{1}{4}68\,481c^2 - 3969c + \tfrac{1215}{4}$$
$$A_2(c) = -14\,325\,712c^9 + 16\,903\,900c^8 - 7706\,160c^7 + 2908\,572c^6$$
$$-50\,780c^5 - 53\,5179c^4 + 65\,382c^3 + 79\,497c^2 - 25\,866c + 2430$$
$$A_3(c) = 19\,873\,632c^8 - 19\,373\,048c^7 + 9076\,800c^6 - 1980\,488c^5$$
$$-1028\,392c^4 + 422\,862c^3 + 180\,036c^2 - 94\,878c + 11\,340$$
$$A_4(c) = -17\,212\,760c^7 + 15\,010\,008c^6 - 5902\,576c^5 - 417\,308c^4$$
$$+1036\,266c^3 + 161\,928c^2 - 212\,058c + 34\,020$$
$$A_5(c) = 10\,051\,680c^6 - 7736\,848c^5 + 1500\,672c^4 + 1175\,928c^3$$
$$-112\,392c^2 - 288\,792c + 68\,040$$
$$A_6(c) = -4046\,656c^5 + 2366\,960c^4 + 430\,656c^3 - 390\,672c^2 - 217\,728c + 90720$$
$$A_7(c) = 1102\,720c^4 - 248\,160c^3 - 317\,376c^2 - 54\,432c + 77\,760$$
$$A_8(c) = -187\,824c^3 - 69\,408c^2 + 33\,264c + 38\,880$$
$$A_9(c) = 15\,936c^2 + 19\,584c + 8640.$$

Now we prove that all $A_0(c), A_1(c), \ldots, A_9(c)$ are positive for $c \leqslant 0$. To this end we calculate the zeros of $A_j(c)$, $j = 0, 1, \ldots, 9$. Using again *Mathematica*, we find:

$A_0(c)$ has only one real zero at $c = 7.592\,59$, hence $A_0(c) > 0$ for $c \leqslant 0$;

$A_1(c)$ has only two real zeros at $c = 0.253\,885$ and $c = 1.502\,57$, hence $A_1(c) > 0$ for $c \leqslant 0$.

Similarly we find that $A_2(c) = 0$ at $0.726\,393$, $A_3(c)$ has no real zeros,

$A_4(c) = 0$ at 0.450 744,
$A_5(c)$ has no real zeros,
$A_6(c) = 0$ at 0.347 347,
$A_7(c)$ has no real zeros,
$A_8(c) = 0$ at 0.572 203, finally $A_9(c)$ has no real zeros.

These calculations show that $A_j(c) > 0$ for $c \leqslant 0$ therefore $Q(c, z) > 0$ for $z \geqslant -c + \frac{1}{2} = |c| + \frac{1}{2}$. By (2.11) we have $\hat{\varphi}^2 = \max\{\tilde{\varphi}^2, -c + \frac{1}{2}\}$ hence

$$\hat{\varphi}^2 = \begin{cases} -c + \frac{1}{2} & \text{for} \quad \frac{-6-\sqrt{78}}{14} < c < 0 \\ \tilde{\varphi}^2 & \text{for} \quad c \leqslant \frac{-6-\sqrt{78}}{14}. \end{cases}$$

Then we have $\tilde{q}(x) < q(x)$ in (2.6) for $x > \hat{a} = f(\hat{\varphi})$, hence application of lemma 2 gives the relations (1.8*a*) and (1.8*b*). □

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